**Particular Solutions of Non-Homogeneous ODEs** Differential Equations X. Du

- There are three main methods to find particular solutions: 1) the following theorem, 2) method of undetermined coefficients, 3) variation of parameters
- Theorem: For an *n*th-order non-homogeneous linear differential equation $y^{(n)} + A(x)y^{(n-1)} + ... + P(x)y' + Q(x)y = f(x)$, let $p(x)$ denote $x^n + A(x)x^{n-1} + ... + xP(x) + Q(x)$. Suppose $f(x) = e^{\alpha x}$, $\alpha \in \mathbb{C}$. Then $y_p = \dfrac{e^{\alpha x}}{p(\alpha)}$, assuming $p(\alpha) \neq 0$.
    - If $f(x)$ is not complex, try to make it complex and undo this step at the end! Example: Let $f(x) = e^x \sin x$. $e^x \sin x = \mathrm{Im}[e^{(1+i)x}]$. Therefore, let $\alpha = 1 + i$. The particular solution is $y_p = \mathrm{Im}\left[\dfrac{e^{\alpha x}}{p(\alpha)}\right]$.
    - If $\alpha$ is a simple root of $p(x)$, i.e. $p(\alpha) = 0$ with multiplicity 1, then $y_p = \dfrac{xe^{\alpha x}}{p'(\alpha)}$.
    - If $\alpha$ is a repeated root of $p(x)$, i.e. $p(\alpha) = 0$, with multiplicity *n*, then $y_p = \dfrac{x^n e^{\alpha x}}{p^{(n)}(\alpha)}$ (idea is similar to L'Hôpital's Rule).
- **Method of Undetermined Coefficients**: Guess something with similar structure to $f(x)$, and then solve with a system of equations.
    - Example: Suppose $f(x) = 3x^2 - 2x + 1$. Guess $y_p = Ax^2 + Bx + C$.
    - Example: Suppose $f(x) = 3\cos 2x$. Guess $y_p = A\cos 2x + B\sin 2x$
    - Example: Suppose $f(x) = 3xe^{2x}$. Guess $y_p = (Ax + B)e^{2x}$
    - However, this doesn't always work. In this case, keep on adding a linear factor of *x* until something works. For example, if the guess $A\cos x + B\sin x$ does not work, then try $A\cos x + B\sin x + Cx\cos x + Dx\sin x$. If that doesn't work, add more *x*'s until it does.
- **Variation of Parameters**: Given an *n*th-order non-homogeneous linear differential equation $y^{(n)} + A(x)y^{(n-1)} + ... + P(x)y' + Q(x)y = f(x)$, let each element of the linearly independent set $\{y_1, y_2, ..., y_n\}$ be a solution to the differential equation. Let $W(x)$ represent the Wronskian determinant of the functions in this set, and let $W_i(x)$ represent the Wronskian determinant with its *i*th column replaced with the vector $\langle 0,0,..., f(x)\rangle$. Then $y_p = \sum_{i=1}^{n} y_i(x) \int \dfrac{W_i(x)}{W(x)} dx$.